SEMISIMPLE FROBENIUS STRUCTURES AT HIGHER GENUS

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To the memory of Tom Wolff

Introduction

The genus g GW-potential of a compact symplectic manifold X is a generating function for genus g Gromov-Witten invariants. It is a formal function

$$F_X^g(t) := \sum_{n=0}^{\infty} \sum_{d\in H_2(X)} \frac{q^d}{n!} \int_{[X_{g,n,d}]} \mathrm{ev}_1^*(t) \wedge ... \wedge \mathrm{ev}_n^*(t),$$

on the cohomology space $H^*(X, \mathbf{Q}\{\{q\}\})$ over a suitable Novikov ring $\mathbf{Q}\{\{q\}\}$. The coefficients are defined by integration over virtual fundamental cycles in the moduli spaces of degree d genus g stable pseudo-holomorphic curves with n marked points. The cohomology classes $\mathrm{ev}_i^*(t)$ are pull-backs from X by the evaluation maps at the marked points.

One may use the natural *contraction maps* $\mathrm{ct} : X_{g,n,d} \to \overline{\mathcal{M}}_{g,n}$ to the Deligne – Mumford moduli spaces of marked Riemann surfaces in order to define more general potentials by integration over inverse images of boundary strata or of any other cycles.

The potentials F_X^g and their generalizations are expected to obey some universal constraints, yet unknown explicitly (see however [4, 12, 13, 22]), but encoded implicitly in the topology of the Deligne-Mumford spaces $\overline{\mathcal{M}}_{g,n}$. In a sense, the implicit constraints, to be considered as axioms of 2-dimensional Topological Field Theory, are the subject of our study in this paper.

In this paper, we will compute genus $g \geq 2$ Gromov-Witten invariants and their generalizations with gravitational descendents in the context of equivariant Gromov-Witten theory of tori actions with isolated fixed points. Both formulas, with and without descendents, are stated in a form applicable to the axiomatic version of genus 0 Gromov – Witten theory, namely — to semisimple Frobenius structures. Therefore the formulas can be considered as definitions extending the genus 0 theory to higher genus in a way consistent — conjecturally — with the implicit axioms mentioned above. In (non-equivariant) Gromov-Witten theory, the formulas become conjectures expressing higher genus GW-invariants in terms of genus 0 GW-invariants of symplectic manifolds with genericly semisimple quantum cup-product.

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1. Definitions and examples

1.1. **Frobenius structures.** The axiomatic structure of 2D TFT is understood well in genus 0 due to R. Dijkgraaf – E. Witten [5], B. Dubrovin [6] and many others (see [23]) as the theory of Frobenius manifolds. By definition, a *Frobenius structure* on a manifold H consists of:
(i) a flat pseudo-Riemannian metric $(\cdot,\cdot)$,
(ii) a function F whose 3-rd covariant derivatives F_{abc} are structure constants $(a \bullet b, c)$ of a Frobenius algebra structure, i.e. associative commutative multiplication $\bullet$ satisfying $(a \bullet b, c) = (a, b \bullet c)$, on the tangent spaces $T_t H$ which depends smoothly on t;
(iii) the vector field of unities $\mathbf{1}$ of the $\bullet$-product which has to be covariantly constant and preserve the multiplication and the metric.

Example 1. The genus 0 GW-potential $F = F_X^0$ defines a Frobenius structure on the super-space $H = H^*(X, \mathbf{Q})$ [1] In this example, the metric and the unit vector field are translation-invariant and defined by the Poincare intersection pairing and by the cohomology class 1 respectively.

Example 2. Let $f(x,t)$, $t \in H$, be a miniversal deformation (with respect to the right equivalence) of the germ $f(\cdot, 0) : (\mathbf{C}^m, 0) \to (\mathbf{C}, 0)$ of a holomorphic function at an isolated critical point. Then the tangent spaces $T_t H$ are canonically identified with the algebras $Q_t := \mathbf{C}\{x\}/(f_x)$ of functions on the critical schemes crit $f(\cdot, t)$ and thus carry a natural multiplication $\bullet$ with unity 1. Let Ω be a holomorphic volume form on $\mathbf{C}^m$ possibly depending on t. The multiplication $\bullet$ is Frobenius with respect to the residue pairing

$$(\phi, \psi) := \frac{1}{(2\pi i)^m} \oint_{|f_{x_1}|=\varepsilon_1} \cdots \oint_{|f_{x_m}|=\varepsilon_m} \frac{\phi(x)\psi(x)\ \Omega}{f_{x_1} \cdots f_{x_m}},$$

which is known to be non-degenerate on Q_t (see [19]). According to the theory [25] of *primitive volume forms* there exists a choice of Ω such that the corresponding residue metric is flat and constitutes, together with the multiplication $\bullet$, a Frobenius structure on H (see also [3] for a new approach).

Frobenius manifolds of Examples 1 and 2 come equipped with one more ingredient — the *Euler vector field* E such that $\bullet$, $\mathbf{1}$ and $(\cdot,\cdot)$ are eigenvectors of the Lie derivative L_E with the eigenvalues 0, -1 and $2 - D$ respectively. Such Frobenius structures are called *conformal*, and D is called their *dimension*. In the Example 1, D coincides with the complex dimension of the target manifold X, and the grading imposed by E originates from grading in cohomology. In Example 2, the Euler vector $E(t)$ is given by the class of the function $f(\cdot, t)$ in the algebra Q_t, and $D = 1 - 2/h$ where h is the so called *Coxeter number* of the singularity [1]. Frobenius manifolds in the next example fall out of the conformal class.

Example 3. Let the Kähler manifold X be endowed with a Hamiltonian Killing action of a compact group T. Then one can introduce *equivariant GW-invariants* [15] using T-equivariant cohomology and intersection theory in the moduli spaces

[1] Formally speaking F_X^0 defines a Frobenius structure over $\mathbf{Q}\{\{q\}\}$. However, due to the divisor equation, 3-rd rderivatives of F_X^0 make sense at $q = 1$ as formal Fourier series along $H^2(X, \mathbf{Q})$ and define a Frobenius structure over $\mathbf{Q}$. We refer to [2, 15, 23] for discussions of these standard subtleties.

$X_{g,n,d}$. The genus 0 equivariant GW-invariants define on $H := H_T^*(X, \mathbf{Q})$ the structure of a Frobenius manifold *over the ground ring* $H^*(BT, \mathbf{Q})$, the coefficient ring of the equivariant cohomology theory. On the other hand, grading in equivariant cohomology imposes homogeneity constraints on GW-potentials so that $(\cdot,\cdot)$, $\mathbf{1}$ and $\bullet$ do have degrees $2 - \dim X$, -1 and 0 with respect to a suitable Euler vector field E. Yet the Frobenius structure is not conformal since elements of the ground ring may have non-zero degrees and therefore L_E is a differentiation only over $\mathbf{Q}$ instead of the ground ring of the Frobenius structure.

A Frobenius manifold is called *semisimple* if the algebras $(T_tH, \bullet)$ are semisimple at generic t. Frobenius structures of Example 1 are semisimple for, say, projective spaces and flag manifolds, and are not semisimple for Calabi-Yau manifolds. Let us assume now on that the group T in Example 3 is a torus acting on X with isolated fixed points only. Then the cup-product in the equivariant cohomology $H_T^*(X, \mathbf{Q})$ is genericly semisimple, resulting in the corresponding Frobenius structure being semisimple too. All Frobenius manifolds of Example 2 are semisimple.

1.2. **The formula.** Our expression for the higher genus potentials F^g of a semisimple Frobenius manifold H has the form

$$\begin{aligned} & e^{\sum_{g\geq 2} \hbar^{g-1} F^g(t)} = \\ & = \left[\, e^{\frac{\hbar}{2}\sum_{k,l=0}^{\infty}\sum_{i,j} V_{kl}^{ij}\Delta_i^{1/2}\Delta_j^{1/2}\partial_{Q_k^i}\partial_{Q_l^j}} \prod_j \tau(\hbar\Delta_j;\ Q_0^j, Q_1^j, \dots) \,\right]_{Q_k^i = T_k^i}, \end{aligned} \tag{1}$$

where $V_{kl}^{ij}, \Delta_j, T_k^i$ are certain functions of $t \in H$ defined at semisimple points, $i, j = 1, \dots, \dim H$, $k, l = 0, 1, 2, \dots$, and τ is the following *Kontsevich – Witten tau-function.*

Let $c^{(1)}, \dots, c^{(n)}$ denote the 1-st Chern classes of the *universal cotangent lines* over the Deligne – Mumford spaces $\overline{\mathcal{M}}_{g,n}$, i.e. line bundles formed by cotangent lines to the curves at the marked points. We put $Q(c) = Q_0 + Q_1c + Q_2c^2 + \dots$ where Q_i are formal variables, introduce the genus g descendent potential of $X = \mathrm{pt}$

$$\mathcal{F}_{\mathrm{pt}}^g(Q) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}} Q(c^{(1)}) \wedge \dots \wedge Q(c^{(n)})$$

and define

$$\tau(\hbar; Q) = \exp\{\sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}_{\mathrm{pt}}^g(Q)\}. \tag{2}$$

As it was proved by M. Kontsevich [20], $\tau(Q)$ provides an asymptotic expansion of the matrix Airy function and (modulo some re-notation) coincides, as it was conjectured by E. Witten [26], with the tau-function of the KdV-hierarchy satisfying the string equation.

In order to define the functions $V_{kl}^{ij}, \Delta_i, T_k^j$ we have to review the structural theory of semisimple Frobenius manifolds [6, 16, 23].

1.3. **Canonical coordinates, Hessians and stationary phase asymptotics.** Given a germ of a Frobenius manifold, we introduce coordinates $\{t^\alpha\}$ flat with respect to the metric $(\cdot,\cdot)$, denote $\{\phi_\alpha\}$ the corresponding frame in the tangent bundle, put $g_{\alpha\beta} := (\phi_\alpha, \phi_\beta)$ and $(g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}$.

The associativity constraint of the $\bullet$-product is expressed by the *WDVV-identity* for the genus 0 potential (we use the summation convention if possible):

$$F_{\alpha\beta\mu}g^{\mu\nu}F_{\nu\gamma\delta} = F_{\alpha\gamma\mu}g^{\mu\nu}F_{\nu\beta\delta} = F_{\alpha\delta\mu}g^{\mu\nu}F_{\nu\beta\gamma}.$$

It can also be interpreted as commutativity of the following connection operators $\nabla_\alpha(z) := z\partial_\alpha + \phi_\alpha \bullet$ on TH and respectively — the compatibility property of the following linear PDE system on T^*H for any value of the parameter $z \neq 0$:

$$z\partial_\alpha S_\beta = F_{\alpha\beta\mu}g^{\mu\nu}S_\nu. \tag{3}$$

The system plays an important role in the theory of Frobenius structures, and we ought to start with some remarks about its solutions. A fundamental solution to (3) can be found in the form of a power z^{-1}-series $1 + z^{-1}S_1 + z^{-2}S_2 + ...$ satisfying the unitary condition $S^*(-1/z)S(1/z) = 1$ (here " $*$" means "adjoint relative to $(\cdot,\cdot)$"). Such a solution S is unique up to right multiplication by a constant matrix $1 + O(z^{-1})$ satisfying the unitary condition. A choice of such a solution is the starting point in Dubrovin's construction [6] of genus 0 gravitational descendents of Frobenius structures. We will review this construction in the section 3 and respectively will make use of such a fundamental solution in our description (22) of higher genus *descendent* potential. However, both higher genus formulas (with or without descendents) require another, asymptorical form of solution to the same system (3) which can be constructed for a *semisimple* Frobenius structure as follows.

Let us assume that the Frobenius manifold is semisimple. In a neighborhood of a semisimple point one introduces *canonical coordinates* $\{u^i(t)\}$ (see [6]). They are characterized uniquely up to reordering and additive constants by the property of $\partial_i := \partial/\partial u^i$ to form the basis of canonical idempotents of the $\bullet$-product on T_tH. The flat metric $(\cdot,\cdot)$ is diagonal in canonical coordinates and is therefore determined by the non-vanishing functions (∂_i, ∂_i). We put $\Delta_i := 1/(\partial_i, \partial_i)$. In singularity theory, u_i are critical values of the Morse functions $f(\cdot, t)$ at the critical points, and Δ_i are the Hessians at these points computed in Ω-unimodular coordinate systems.

Let U denote the diagonal matrix of canonical coordinates $\operatorname{diag}(u_1, ..., u_N)$, and Ψ denote the transition matrix between the flat and normalized canonical bases: $\Delta_i^{-1/2}du^i = \sum_\beta \Psi^i_\beta dt^\beta$. In particular, $\sum_i \Psi^i_\alpha \Psi^i_\beta = g_{\alpha\beta}$, $\Psi^i_\mu g^{\mu\nu}\Psi^j_\nu = \delta_{ij}$.

Proposition (see [6, 16]).

(a) Near a semisimple point the system (3) has a fundamental solution in the form of the matrix series :

$$S = \Psi(R_0 + zR_1 + z^2R_2 + ...) \exp U/z \tag{4}$$

where $R_k = (R_k)_i^j$ are matrix-functions of u, and $R_0 = 1$.

(b) The series solution S can be chosen to satisfy the unitary condition

$$(1 + zR_1 + z^2R_2 + ...)(1 - zR_1^t + z^2R_2^t - ...) = 1 \tag{5}$$

(c) The series $R = 1 + zR_1 + z^2R_2 + ...$ in the solution S satisfying the unitary condition is unique up to right multiplication by unitary diagonal matrices $\exp(a_1z + a_2z^3 + a_3z^5 + ...)$ *where $a_k = \operatorname{diag}(a_k^1, ..., a_k^N)$ are constant.*

(d) In the case of conformal Frobenius structures the series R in a fundamental solution S can be chosen homogeneous, and such R is unique and possesses the unitary property automatically.

Proof. A proof of (d) and (a) is given in [6] and [16]. We will remind below some details from [16] in order to justify the additions (b) and (c) needed here.

Substitution of $S = \Psi(1 + ...) \exp(U/z)$ into (3) yields a chain of equations $(d + W\wedge)R_{k-1} = [dU, R_k]$, where $W = \Psi^{-1}d\Psi = [dU, R_1]$, to be solved inductively starting with $R_0 = 1$. First, off-diagonal entries of R_k are expressed algebraically via R_{k-1}, then the diagonal terms of R_k are found by integration from the next equation using the fact that $[dU, R_{k+1}]$ has zero diagonal entries. Compatibility conditions needed in this procedure are verified in [16].

In order to prove (b), let us introduce a temporary notation $P_k = R_k R_0^t - R_{k-1}R_1^t + ...(-1)^k R_0 R_k^t$ for the z^k-term in $R(z)R^t(-z) = 1 + P_1 z + P_2 z^2 + ...$ A short elementary computation shows that $[dU, P_k] = dP_{k-1} + [W, P_{k-1}]$. Assuming that $P_{k-1} = 0$ (or 1 for $k = 0$), we conclude that off-diagonal entries of P_k vanish. This already implies $P_k = 0$ for odd k since such P_k are obviously anti-symmetric. Now, taking in account that P_k is diagonal and $W = \Psi^{-1}d\Psi$ is anti-symmetric, we conclude from the next equation $dP_k + [W, P_k] = [dU, P_{k+1}]$ that the diagonal entries of P_k are constant. For even k we have $P_k = R_k + R_k^t + ...$ and thus a unique choice of integration constants in the above procedure for finding R_k will make P_k vanish.

Yet the integration constants for diagonal entries of R_{2k-1} are totally ambiguous, and it is immediate to see, by induction on k, that this ambiguity is correctly accounted by the multiplication $R \mapsto R\exp(a_k z^{2k-1})$ described in (c).

In the conformal case, let $E = \sum u^i \partial_i$ denote the Euler field. The Euler formula $R_k = -(i_E dR_k)/k$ shows how to recover diagonal entries of R_k via their differentials by an algebraic procedure. This implies existence of a homogeneous solution R. Finally, the homogeneity condition leaves no freedom in the choice of the integration constants, but it also guarantees that the constant diagonal entries in P_{2k} are zeroes. This proves (d).

Let $S(z)$ be the unitary fundamental solution to (3) singled-out in the proposition. We introduce a new matrix-function

$$[V^{ij}(z,w)] := (z+w)^{-1}[S^i_\mu(z)]^t[g^{\mu\nu}][S^j_\nu(w)].$$

It expands as $V^{ij}(z,w) =$

$$\frac{e^{u^i/z+u^j/w}}{z+w}\sum_s R^i_s(z)R^j_s(w) =: e^{u^i/z+u^j/w}\left(\frac{\delta^{ij}}{z+w} + \sum_{k,l=0}^{\infty}(-1)^{k+l}V^{ij}_{kl}z^k w^l\right). \tag{6}$$

This defines V^{ij}_{kl} as functions on the Frobenius manifold in a neighborhood of a semisimple point.

Next, in the semisimple Frobenius algebras $(T_t H, \bullet)$ we have :

$$1 = \sum \delta^\mu \phi_\mu = \sum \partial_j = \sum \Delta_j^{-1/2}(\Delta_j^{1/2}\partial_j).$$

We expand "the first row" of $S(z)$

$$\sum \delta^\mu S^i_\mu(z) = \Big(\sum_j \Delta_j^{-1/2}R^i_j(z)\Big)e^{u^i/z} =: \Big[1 - \sum_{k=0}^{\infty} T^i_k(-z)^{k-1}\Big]\frac{e^{u^i/z}}{\sqrt{\Delta_i}}. \tag{7}$$

This defines T^i_k. In particular, $T^i_0 = T^i_1 = 0$.

Perhaps, the nature of these formulas, the asymptotical solution $S(z)$ and the relationship between the two forms of fundamental solutions to (3) will become more transparent after the following two examples.

Example 4. In singularity theory, a fundamental solution matrix to the equation (3) is given by complex oscillating integrals of suitable m-forms over suitable m-cycles :

$$S^i_\mu = \int_{\Gamma^i \subset \mathbf{C}^m} e^{f(x,t)/z} \phi_\mu(x,t)\Omega \ .$$

The cycles Γ^i can be constructed as in Morse theory for the function $\mathrm{Re}\{f(\cdot,t)/z\}$ and thus correspond to critical points $x^i(t)$ of the function $f(\cdot,t)$. To construct $1/z$-expansion of the integrals, one first expands the integrals over the levels $f = f_{crit} - \tau$ near $\tau = \infty$ and then describes the oscillating integrals via the Laplace transform. Say, for weighted-homeogeneous singularities

$$\int_{\gamma_i} \phi_\mu(x,t) \frac{\Omega}{df(x,t)} = \tau^{d_\mu} (\sum A^i_{k,\mu}(t)\tau^{-k}),$$

where d_μ is the weight of the form $\phi_\mu \Omega/df$. Respectively,

$$S^i_\mu = \sum A^i_{k,\mu} \int_0^\infty e^{-\tau/z} \tau^{d_\mu - k} \ d\tau = z^{d_\mu + 1} \sum A^i_{k,\mu}(t) \Gamma(d_\mu + 1 - k) z^{-k}.$$

Alternatively, one arrives to the expansion (4) via the stationary phase asymptotics of the oscillating integrals near non-degenerate critical points of Morse functions $f(\cdot,t)$:

$$\int_{\Gamma^i} e^{f(x,t)/z} \phi_\mu(x,t) \ \Omega \sim e^{u^i/z} (\frac{\phi_\mu(x^i,t)}{\sqrt{\Delta_i}} + ...)$$

where $u^i = f(x^i,t)$ is the critical value and Δ_i is the Ω-Hessian at the critical point. In particular (7) is the stationary phase expansion

$$\int_{\Gamma^i} e^{f/z}\Omega \sim \frac{e^{u^i/z}}{\sqrt{\Delta_i}} [1 + T^i_2 z - T^i_3 z^2 + ...]$$

Example 5. In Gromov-Witten theory, a $1/z$-series solution to (3) satisfying the unitary condition is given by the following matrix of gravitational *descendents*:

$$(8) \quad \langle \phi_\beta, \frac{\phi_\gamma}{z-c} \rangle := \sum_{n,d} \frac{q^d}{n!} \int_{[X_{0,2+n,d}]} \mathrm{ev}_0^*(\phi_\beta) \wedge \mathrm{ev}_1^*(t) \wedge ... \wedge \mathrm{ev}_n^*(t) \wedge \frac{\mathrm{ev}^*_{n+1}(\phi_\gamma)}{z - c^{(n+1)}} \ .$$

By definition, the constant $g_{\alpha\beta}$ is taken on the role of the ill-defined term with $d = 0, n = 0$. This solution is related to the *two-point descendent*

(9)

$$\langle \frac{\phi_\alpha}{z-c}, \frac{\phi_\beta}{w-c} \rangle := \sum_{n,d} \frac{q^d}{n!} \int_{[X_{0,2+n,d}]} \frac{\mathrm{ev}_0^*(\phi_\alpha)}{z - c^{(0)}} \wedge \mathrm{ev}_1^*(t) \wedge ... \wedge \mathrm{ev}_n^*(t) \wedge \frac{\mathrm{ev}^*_{n+1}(\phi_\beta)}{w - c^{(n+1)}}.$$

in the same way as $S(z)$ is related to $V(z,w)$:

$$(10) \qquad \langle \frac{\phi_\alpha}{z-c}, \frac{\phi_\beta}{w-c} \rangle = \sum_{\mu\nu} \langle \frac{\phi_\alpha}{z-c}, \phi_\mu \rangle g^{\mu\nu} \langle \phi_\nu, \frac{\phi_\beta}{z-c} \rangle.$$

According to the mirror conjecture [14, 16] the descendents (8) can be identified with oscillating integrals of the mirror partner. When this is the case the values of

Δ^i and T_k^i can be extracted from the stationary phase asymptotics of the integrals. In Section 2 (see also [16]), we will find that in the equivariant setting of Example 3 when the fixed points of the toris action are isolated and respectively the classical equivariant cohomology algebra of the target space is semisimple, the series $S(z)$ and $V(z,w)$ essentially coincide with the descendents (8) and (9).

Conjecture 1. *With the notations (2,6,7) in force, the formula (1) represents higher genus GW-invariants of compact symplectic manifolds with generically semisimple quantum cup-product.*

The main reason to believe in the conjectural formula (1) is the theorem in the next section and the miraculous coincidences which occur in the proof. We would like to mention here one more bit of evidence in its favor. Namely, in the case of conformal Frobenius manifolds of dimension $N = 2$ ("two primaries") our formula yields, after some computation, the following genus 2 potential:

$$\frac{d(3d-1)(d-1)^2(3d-5)(d-2)}{2880}\,\frac{\Delta_-}{(u^+-u^-)^3},$$

where d is the conformal dimension, and $u^{\pm}$ are the canonical coordinates. This answer coincides with the result found in [9].

Note that the potential vanishes in the case $d = 1/3$ corresponding to the singularity of type A_2. This fact agrees with the general conjecture that our formula (1), when applied to the Frobenius structures on the miniversal deformations of isolated critical points, should give rise to higher genus potentials which extend analytically through the bifurcation set (and therefore must vanish for A, D, E-singularities). For no apparent reason, the above formula is symmetric about $d = 1$ (corresponding to GW-invariants of $\mathbf{C}P^1$). It would be interesting to find out what is behind this symmetry.

2. Computation in equivariant GW-theory

In this section, we formulate and prove Theorem 1 confirming the conjectural formula (1) in the case of equivariant Gromov – Witten invariants of Hamiltonian tori actions with isolated fixed points. Roughly speaking, we will compute the GW-invariants using fixed point localization and will see how the formula (1) emerges from the combinatorial formalism of summation over graphs. We will first discard those factors in localization formulas which are due to the so called Hodge intersection numbers. This will lead us to a (wrong!) higher genus potential formula based on a matrix series $R(z)$ corresponding to some reference choice $a_k^i = 0$ of the integration constants of the part (c) of Proposition. Then we will point out a new choice of the integration constants a_k^i which compensates for the effect of the Hodge integrals and yields a right formula for the higher genus potential.

2.1. **Localization and materialization.** Let the torus T act on X with isolated fixed points only. Fixed points of the induced action of T on the moduli spaces $X_{g,n,d}$ can be described as curves formed by *legs* — 1-dimensional orbits of $T_{\mathbf{C}}$ in X or their multiple covers, — which are connected at *joints* — nodes or DM-stable curves mapped to fixed points X^T. Due to multiplicative properties of the Euler classes contributions of fixed points into localization formulas essentially factors into contributions of legs and joints [21, 18]. (We are assuming for simplicity that the 1-dimensional orbits are also isolated. Beyond this assumption, our arguments

remain valid but the leg contributions are to be found by integration over suitable orbit spaces.) Contributions of fixed point submanifolds can be arranged as the sum over strata in Deligne – Mumford spaces in accordance with images of the submanifolds under the contraction map ct : $X^T_{g,n,d} \to \overline{\mathcal{M}}_{g,0}$. It is convenient to name some elements of T-invariant curves depending on their fate under the contraction map. We call *vertices* those joints of T-invariant curves in X which contract to irreducible components of DM-stable $(g,0)$-curves. The genus 0 trees of legs and joints which contract to (self-)intersection points of these components are called *edges*. The trees which contract to non-singular points are called *tails*.

Thinking of a T-invariant curve (may be disconnected) as a collection of vertices (DM-stable curves mapped to the fixed points X^T) with arbitrary number of tails attached and connected somehow by the edges, we arrive at the fixed point expression for the higher genus potential with the standard combinatorics (1) of Wick's formula. Contributions of vertices will be expressible via intersection numbers (2) in Deligne – Mumford spaces, while the edge factors and tail factors should be extracted from genus 0 GW-invariants of X.

A key point is that *the genus* 0 *data needed in the localization formulas can be written in abstract terms of semisimple Frobenius structures, and vice versa.* For example, in the GW-theory of X, the sum $\sum u^i$ of canonical coordinates enumerates elliptic curves with a *fixed complex structure*. Expressing the GW-invariant via the sum over fixed point components we can single out the sub-sum where *the* elliptic joint of the curve is mapped to the i-th fixed point in X. It turns out [15] that the sub-sum equals u_i. Another example: let $\{\phi_\alpha\}$ be the basis of δ-functions at the fixed points in localization of $H^*_T(X)$, so that $g^{\alpha\beta} = e_\alpha \delta_{\alpha\beta}$ where $\sum e_\alpha \phi_\alpha$ is the equivariant Euler class of TX. In the fixed point sum for $F^{(0)}_{\alpha\alpha\alpha} e^{3/2}_\alpha$ (no summation) we single out contributions with the three distinguished marked points belonging to the same joint of the curve. The sub-sum turns out to coincide with $\Delta^{1/2}_\alpha$. We refer to [15, 16] for further details of this *materialization* phenomenon in the theory of canonical coordinates. Our computation of higher genus potentials uses some of these results along with the standard fixed point localization technique [21, 18] in the moduli spaces of stable maps.

The edge factors mentioned earlier are identified with V^{ij}_{kl}. First, in the fixed point expression for $e_i \langle \frac{\phi_i}{\chi - c}, \frac{\phi_i}{z-c} \rangle$ we single out contributions of those fixed point where the first and the last marked points belong to the same joint of the curve. The sum of such contributions turns out to coincide with $e^{u^i(1/\chi + 1/z)}/(z+\chi)$ (see [16, 14]). Therefore this expression occurs in the localization formula for the one-point descendent $\langle \phi_\alpha, \phi_i/(z-c) \rangle$ as the factor responsible for the contributions of the joints carrying the last marked point. The variable χ is to be replaced by the character of the torus action on the leg approaching the joint from the direction of the first marked point. Thus the dependence of the descendent on z is transparent from the expansion of the factor: $e^{u^i/z}[\sum e^{u^i/\chi}(-z)^k/\chi^{k+1}]$. We conclude that the matrix $[\ \langle \phi_\alpha, \frac{\phi_j}{z-c} \rangle \sqrt{e_j}\ ]$ (normalized this way) is a unitary solution S of the part (b) of Proposition. It is one of the solutions described by the part (b) of Proposition. Among the total class of solutions (see part (c)), it is characterized by the property that the series $R(z)$ turns into 1 in the limit of classical equivariant cohomology, that is when contributions of all non-constant stable maps are neglected. Eventually we will have to change this normalization of the solution S in order to compensate the effect of Hodge integrals in localization formulas.

Processing similarly contributions of the joints carrying the first and last marked points in localization formulas for the two-point descendent (9), we extract the edge factors mentioned above:

$$(11)\quad \langle \frac{\phi_i}{z-c}, \frac{\phi_j}{w-c} \rangle \sqrt{e_i e_j} = e^{u^i/z+u^j/w}[\frac{\delta_{ij}}{z+w} + \sum (-z)^k(-w)^l \text{ (edge factor)}^{ij}_{kl}].$$

Taking into account (6) and (10) we conclude that the edge factors are identified with the coefficients V^{ij}_{kl} corresponding to the solution S. Note that the weights $e^{u^j/\chi}$ are incorporated into the edge factors.

Computing contributions of vertices, denote by χ^i_r, $r = 1, ..., \dim_{\mathbf{C}} X$, the characters of the torus action on the tangent space to X at the fixed point with the index i. The localization formulas require the following intersection numbers in the Deligne-Mumford spaces:

$$(12)\qquad \sum_{n=0}^{\infty} \frac{e_i^{-1}}{n!} \int_{\overline{\mathcal{M}}_{g,m+n}} \frac{\prod_{s=1}^{g} \prod_r (\chi^i_r - \rho_s)}{(x_1 - c^{(1)})...(x_m - c^{(m)})} \wedge Q(c^{(m+1)}) \wedge ... \wedge Q(c^{(m+n)}).$$

Here $\rho_1, ..., \rho_g$ are Chern roots of the *Hodge bundle* with the fiber $H^1(\Sigma, \mathcal{O}_\Sigma)^*$, and $x_1, ..., x_m$ are formal variables. In localization formulas, these variables are replaced by some χ^i_r (or their fractions), the characters of the torus action on the m edges adjacent to the vertex. The formula (1) accounts for this substitution by matching the factors $c^k x^{-k-1}$ in (12) with the corresponding edge factors $V^{i\cdots}_{k\cdots}$ in (11).

The series $Q(c) = Q^i_0 + Q^i_1 c + ...$ is to be substituted in the localization formulas by the localization factor of the tail approaching the i-th fixed point, and the next task is to interpret the factor in terms of abstract Frobenius structures. For this, we notice that the same series $Q(c)$ occurs — in the same role — in fixed point localization of genus 0 invariants. In particular, the one-point descendent $\langle \frac{\phi_i}{z-c} \rangle = z\langle 1, \frac{\phi_i}{z-c} \rangle$ is written as

$$\frac{z}{e_i} + \frac{Q(-z) - Q(0)}{e_i} + \sum_{n=2}^{\infty} \frac{1}{n!} \int_{\overline{\mathcal{M}}_{0,1+n}} Q(c^{(1)}) \wedge ... \wedge Q(c^{(n)}) \wedge \text{ev}^*_{n+1} \frac{\phi_i}{z-c}.$$

When $Q^i_0 = 0$, it coincides with $(z + Q(-z))/e_i$. This can be achieved by moving the series Q by the *string flow*, and the time needed in order to make $Q^i_0 = 0$ is exactly $-u_i$ (see [15], Section 12, or [16]). Furthermore, both the descendent and the potential (12) are eigenfunctions of the string operator $\partial/\partial Q_0 - \sum Q_{k+1}\partial/\partial Q_k$ (with the eigenvalues $1/z$ and $1/x_1 + ... + 1/x_m$) and of the *dilaton operator* $\partial/\partial Q_1 - \sum Q_k \partial/\partial Q_k$ (with the eigenvalues -1 and $2g - 2 + m$ respectively). Thus, moving along the string flow during the time interval $-u^i$ and then along the dilaton flow during the time interval $\ln \sqrt{\Delta_i}$ we make Q^i_0 and Q^i_1 vanish and find the final values $Q^i_k = T^i_k$ from (7). The toll to pay consists of the factor $\Delta_i^{g-1+m/2}$ distributed in (1) among vertices and edges and the weights $\exp u^i/\chi^i_r$ already incorporated, as we remarked earlier, into $V^{i\cdots}_{k\cdots}$.

2.2. **Compensating constants.** Yet, with our current definition of V^{ij}_{kl} and T^i_k the formula (1) would represent correctly the fixed point localization of higher genus potentials only if the Hodge factors in (12) were replaced with the factor $\prod_{s,r} \chi^i_r = e^g_i$ (which cancels with other occurrences of e_i here and there). The Hodge factors should be digested as follows. Let N_k denote Newton symmetric

polynomials. It is known [11] that $N_{2k}(\rho)=0$. We rewrite

$$\prod_{s,r}(\chi_r^i-\rho_s)=e_i^g\exp[-\sum_{k=1}^{\infty}N_{2k-1}(1/\chi^i)N_{2k-1}(\rho)/(2k-1)].$$

Let us redefine the fundamental solution $S=[\langle\phi_\alpha,\phi_i/(z-c)\rangle e_i^{1/2}]$ using the ambiguity described in the part (c) of Proposition:

$$^{new}S^i_\alpha := S^i_\alpha \exp[-\sum z^{2k-1}\frac{N_{2k-1}(1/\chi^i)}{2k-1}\frac{B_{2k}}{2k}].$$

Here B_{2k} denote Bernoulli numbers, $z/(\exp z-1)=1-z/2+\sum_k z^{2k}B_{2k}/(2k)!$ The coefficients V^{ij}_{kl} in (6) are redefined accordingly.

Theorem 1. *In equivariant Gromov – Witten theory for Hamiltonian tori actions with isolated fixed points, we obtain the higher genus potential in the form (1) by taking* ^{new}S *on the role of the fundamental solution* S *in (6) and (7).*

Remark. According to the Proposition, the unitary solution ^{new}S is charachterized by the condition that the corresponding series $^{new}R(z)$ turns into the diagonal matrix of the compensating constants $\exp[-\sum z^{2k-1}\frac{N_{2k-1}(1/\chi^i)}{2k-1}\frac{B_{2k}}{2k}]$ in the limit of *classical* equivariant cohomology. Thus Theorem 1, under its hypotheses, coincides with Conjecture 1 where the solution S defined on the basis of Proposition is normalized in this particular way.

Example 6. In genus 1, the differential of the GW-potential was computed by fixed point localization in [16]. In our current notation

$$dF^1_X=\sum_i[\ \frac{V^{ii}_{00}}{2}du^i-\frac{N_1(1/\chi^i)}{24}du^i+\frac{d\Delta_i}{48\Delta_i}\ ].$$

The first summand represents contributions of cycles of rational curves, that is of graphs with one vertex (of type $(g,m)=(0,3)$) and one edge. The other two summands come from (12) with $(g,m)=(1,1)$. The middle term is due to the Hodge integral $\int_{\overline{\mathcal{M}}_{1,1}}\sum\rho_s=1/24$. It can be interpreted as contributions of cycles of rational curves shrinking to a point and is incorporated into the first term as $^{new}V^{ii}_{00}=V^{ii}_{00}-N_1(1/\chi^i)/12$. This change of notation agrees with the theorem since $B_2/2=1/12$. We arrive at the conjecture [16] making sense for arbitrary semisimple Frobenius manifolds:

$$dF^1=\sum_i[\ \frac{V^{ii}_{00}}{2}du^i+\frac{d\Delta_i}{48\Delta_i}\ ].$$

In the case of conformal Frobenius structures the conjecture was proved in [7] by showing that this is the only homogeneous formula that agrees with Getzler's equation [12].

2.3. **Hodge intersection numbers.** We have already explained why the formula (1) for higher genus potentials would arise if the Hodge factors in the vertex contributions (12) were neglected. To derive the theorem it remains to prove that the effect of Hodge factors is correctly accounted by the modification $S\mapsto\ ^{new}S$. For

this, let us introduce the generating function for Hodge intersection numbers:

$$\lambda(\hbar;Q;s_1,s_2,...) = \exp\{\sum_{g=0}^{\infty}\hbar^{g-1}\mathcal{H}_{pt}^{g,n}(Q,s_1,s_2,...)\} \tag{13}$$

where

$$\mathcal{H}_{pt}^{g,n} := \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\overline{\mathcal{M}}_{g,n}} Q(c^{(1)})\wedge ...\wedge Q(c^{(n)})\wedge e^{\sum s_k N_{2k-1}(\rho)/(2k-1)!}.$$

We can introduce a family of fake higher genus potentials depending on the parameters $\{s_k^i\}$ by replacing the factors $\tau(\hbar\Delta_i;Q^i)$ in (1) with $\lambda(\hbar\Delta_i;Q^i;s_1^i,s_2^i,...)$. The actual higher genus potential corresponds to $s_k^i = -(2k-2)!\ N_{2k-1}(1/\chi^i)$. We claim that *the s-parametric deformation of (1) is identified with the a-parametric deformation of the fundamental solution S described in the part (c) of Proposition by taking $a_k^i = B_{2k}s_k^i/(2k)!$* This obviously implies the Theorem.

Following (6) and (7) with scalar $R(z) = \exp(a_1 z + a_2 z^3 + ...)$ and $\Delta = 1$, we introduce the operator $P(a_1,a_2,...) = \frac{1}{2}\sum v_{kl}(a)\partial_{\tilde{Q}_k}\partial_{\tilde{Q}_l}$ where

$$\frac{1}{z+w} + \sum v_{kl}(a_1,a_2,...)\ (-z)^k(-w)^l := \frac{\exp\{\sum a_k(z^{2k-1}+w^{2k-1})\}}{z+w} \tag{14}$$

and define a substitution $\tilde{Q}(Q,s)$ by

$$z+\tilde{Q}(-z) := [z+Q(-z)]\exp[\sum a_k z^{2k-1}]. \tag{15}$$

Lemma.

$$\lambda(\hbar;Q;s_1,s_2,...) = [e^{\hbar P(\frac{B_2}{2!}s_1,\frac{B_4}{4!}s_2,...)}\tau(\hbar;\tilde{Q})]_{\tilde{Q}=\tilde{Q}(Q,\frac{B_2}{2!}s_1,\frac{B_4}{4!}s_2,...)}$$

Our claim follows formally from Lemma. Indeed, the a-parametric modification ${}^{new}S = S\exp(\sum a_k z^{2k-1})$ of the fundamental solution affects the values $Q_k^i = T_k^i$ by some linear transformation and also changes coefficients V_{kl}^{ij} of the differential operator in the exponent of (1). Instead of changing the values T_k^i one can make the change of the variables $Q^i \mapsto \tilde{Q}^i$ and leave the values $Q_k^i = T_k^i$ unchanged. The change of variables coincides with (15). The same change of variables in the differential operator accounts for the most of the change in the coefficients V_{kl}^{ij}. The only remaining discrepancy comes from the term $\delta_{ij}/(z+w)$ in (6) and is determined by (14) as ${}^{new}V_{kl}^{ij} = V_{kl}^{ij} + \delta_{ij}v_{kl}(a_1^i,a_2^i,...)$. Thus $\sum_i \hbar\Delta_i P(a^i)$ is added to the differential operator in the exponent of (1). According to Lemma the modification is equivalent to using $\lambda(\hbar;Q;s)$'s (instead of $\tau(\hbar;Q)$ in (1)) when $a_k^i = B_{2k}s_k^i/(2k)!$

Proof of the lemma. It is known [10], at least in principle, how to compute λ in terms of τ using Mumford's Grothendieck - Riemann - Roch formula [24] for the Chern character $-\sum N_{2k-1}(\rho)/(2k-1)!$ of the Hodge bundle. Moreover, the formula is interpreted in [11] as the PDE-system

$$\frac{\partial}{\partial s_m}\lambda = \frac{B_{2m}}{(2m)!}(\hbar D_m + L_m)\lambda,\ m=1,2,... \tag{16}$$

where $D_m := \frac{1}{2}\sum_{k+l=2m-2}(-1)^k\partial_{Q_k}\partial_{Q_l}$ and $L_m := \partial_{Q_{2m}} - \sum_{k=0}^{\infty}Q_k\partial_{Q_{k+2m-1}}$. The operators $\hbar D_m + L_m$ commute pairwise. The vector fields L_m on the space of power series $Q(c) = Q_0 + Q_1 c + ...$ are linear with respect to the origin shifted to c. In fact they are given by the operators of multiplication by $-c^{2m-1}$. Therefore

L_m commute themselves and define the flow (15). Furthermore, for functions $f(\tilde{Q})$ we find by differentiation that $[\frac{\partial}{\partial a_m}(Pf)](\tilde{Q}(Q,a)) = D_m[f(\tilde{Q}(Q,a)]$. The lemma follows: both sides satisfy the same PDE system (16) and coincide at $s=0$.

3. Generalization to gravitational descendents

The genus g descendent GW-potential of X is a formal function on the *space of curves* $\mathbf{t} = t_0 + t_1 c + t_2 c^2 + ...$ in H defined by

$$\mathcal{F}^g_X(\mathbf{t}) := \sum_{n,d} \frac{q^d}{n!} \int_{[X_{g,n,d}]} \mathrm{ev}_1^* \, \mathbf{t}(c^{(1)}) \wedge ... \wedge \mathrm{ev}_n^* \, \mathbf{t}(c^{(n)}). \tag{17}$$

Here $c^{(i)}$ is the 1-st Chern class of the universal cotangent line over $X_{g,n,d}$ at the i-th marked point, and ev_i^* acts on coefficients t_m of the series $\mathbf{t}$. We present here a conjectural formula for higher genus descendent potentials that makes sense for arbitrary semisimple Frobenius structures. For this, we have to review the construction [6] of genus 0 descendents of Frobenius manifolds.

3.1. **Descendents in genus 0.**

One starts with a fundamental solution $1+z^{-1}S_1+z^{-2}S_2+...$ to the system (3) satisfying the unitary condition and takes it on the role of the one-point descendent (8):

$$(\langle \phi_\alpha, \phi_\mu/(z-c)\rangle g^{\mu\beta}) := 1 + \sum_{k>0} z^{-k} (\langle \phi_\alpha, \phi_\mu c^{k-1}\rangle' g^{\mu\beta}) := 1 + \sum_{k>0} z^{-k} S_k.$$

We emphasize that the $1/z$-series solution is considered disjoint from the asymptotical solution $S = \Psi R(z) \exp(U/z)$ of the Proposition. In particular, in equivariant GW-theory the one-point descendent correlators, defined intrinsicly, form the fundamental solution series in question, and this definition is not affected by the modification $R \mapsto {}^{new}R$ of integration constants in the series R.

Next, one introduces the 2-point descendent (9) using (10):

$$\langle \frac{\phi_\alpha}{z-c}, \frac{\phi_\beta}{w-c}\rangle = \frac{g_{\alpha\beta}}{z+w} + \sum \frac{\langle \phi_\alpha c^m, \phi_\beta c^l\rangle'}{z^{m+1}w^{l+1}} := \langle \phi_\mu, \frac{\phi_\alpha}{z-c}\rangle \frac{g^{\mu\nu}}{z+w} \langle \phi_\nu, \frac{\phi_\beta}{w-c}\rangle. \tag{18}$$

The singular term is present to make the sum satisfy the string equation but it makes the symbol $\langle\cdot,\cdot\rangle$ not entirely bilinear. We use here the notation $\langle\cdot,\cdot\rangle'$ for the honest bilinear 2-point descendents.

Furthermore, one considers the map

$$\mathbf{t} \mapsto t(\mathbf{t}) = crit \ \langle \mathbf{t}(c) - c, 1\rangle(t) \tag{19}$$

from the curve space to the Frobenius manifold defined by taking the critical point of the function $\langle \mathbf{t}(c)-c, 1\rangle := (t_0, t) + \langle \mathbf{t}(c) - c, 1\rangle'$ of $t \in H$ depending linearly on the parameter $\mathbf{t} = t_0 + t_1 c + ...$ One can show that the equation of the critical point takes on the form $t^\alpha = t_0^\alpha + g^{\alpha\mu}\langle \phi_\mu, (\mathbf{t}(c) - \mathbf{t}(0))/c\rangle(t)$ and thus admits a unique formal solution which turns into $t = t_0$ when $t_1 = t_2 = ... = 0$. Finally one puts

$$\mathcal{F}^0(\mathbf{t}) = \frac{1}{2}\langle \mathbf{t}(c) - c, \mathbf{t}(c) - c\rangle'(t(\mathbf{t})) \tag{20}$$

As it is shown in [6], the formula (20) agrees with the *string equation* and the genus 0 *topological recursion relation* and is the only deformation of $\mathcal{F}^0|_{t_1=t_2=...=0} =$

$F^0(t_0)$ satisfying these conditions. Also, (20) agrees with the *dilaton equation* and is consistent with the definition (18):

$$\partial_{t_m^\alpha}\mathcal{F}^0(\mathbf{t}) = \langle\phi_\alpha c^m, \mathbf{t}(c) - c\rangle'(t(\mathbf{t})),\ \partial_{t_m^\alpha}\partial_{t_l^\beta}\mathcal{F}^0(\mathbf{t}) = \langle\phi_\alpha c^m, \phi_\beta c^l\rangle'(t(\mathbf{t})). \tag{21}$$

We emphasize that all the two-point descendent correlators $\partial_{t_m^\alpha}\partial_{t_l^\beta}\mathcal{F}^0$ depend on the infinitely many variables $\mathbf{t}$ only via the substitution $t(\mathbf{t})$.

3.2. **Descendents in higher genus.** Our proposal for higher genus descendent potential has the same form as (1):

$$e^{\sum_{g\geq 2}\hbar^{g-1}\mathcal{F}^g(\mathbf{t})} = [\ e^{\frac{\hbar}{2}\sum \mathbf{V}_{kl}^{ij}\sqrt{\mathbf{D}_i\mathbf{D}_j}\partial_{Q_k^i}\partial_{Q_l^j}} \prod_j \tau(\hbar\mathbf{D}_j; Q^j)\ ]_{Q_k^i=\mathbf{T}_k^i}\ . \tag{22}$$

The functions $\mathbf{V}_{kl}^{ij}, \mathbf{D}_i, \mathbf{T}_k^i$ on the curve space are defined near a semisimple point $\mathbf{t}(0)$ in terms of genus 0 descendents. The definitions are motivated by computation of higher genus descendent potentials in equivariant GW-theory in the presence of the torus acting on the target space with isolated fixed points. The result of the computation is stated in Theorem 2 below. The proof of Theorem 2 follows is identical to the proof of Theorem 1 given in Section 2, with one deviation which is discussed presently. In the part of the text preceeding the formulation of Theorem 2 we assume that the reader is familiar with the details of Section 2.

Let us remind from Section 2 that the formula (22) with the combinatorial structure of a graph sum originates from the technique of fixed point localization in moduli spaces of stable maps, and that the edge and tail factors are to be extracted from the genus 0 descendents. In particular the edge factors in localization formulas are extracted from the expansion (11) for 2-point correlators on the curve space. Due to (21) all such 2-point correlators coincide with the corresponding 2-point descendents on H lifted to the curve space by the change of variables (19). This also applies to $u^i = u^i(t(\mathbf{t}))$ (which can be described via 2-point correlators, see [15, 16]) and therefore — to the edge factors:

$$\mathbf{V}_{kl}^{ij}(\mathbf{t}) = V_{kl}^{ij}(t(\mathbf{t})) \text{ where } t(\mathbf{t}) \text{ is defined by (19).} \tag{23}$$

We stress that in the present context of fixed point localization the edge factors V_{kl}^{ij} will eventually have to be the same as in Theorem 1 (i. e. based on the solution ${}^{new}S$ modified by the Bernoulli constants) in order to compensate the effect of Hodge integrals.

Similarly, the functions $\mathbf{D}_i$ and $\mathbf{T}_k^i$ in the localization formulas are found from the expansion of the 1-point correlator on the curve space:

$$\sqrt{e_i}\ \langle\frac{\phi_i}{z-c}\rangle(\mathbf{t}) = \frac{e^{u^i/z}}{\sqrt{\mathbf{D}_i}}(z + \sum \mathbf{T}_k^i(-z)^k).$$

However — and this is the point that makes the difference — $\langle\frac{\phi_i}{z-c}\rangle$ no longer coincides with the 2-point correlator $z\langle 1, \frac{\phi_i}{z-c}\rangle$ and respectively $\mathbf{D}_i$ and $\mathbf{T}_k^i$ are not obtained from Δ_i and T_k^i by the substitution $t(\mathbf{t})$. We need (18–21) in order to interpret them in terms of abstract Frobenius structures. Let us recall that the asymptotical solution $S_\mu^i(z)$ in the context of fixed point localization actually coincides with $\langle\phi_\mu, \frac{\phi_i}{z-c}\rangle\sqrt{e_i}$. We have

$$\sqrt{e_i}\langle\frac{\phi_i}{z-c}\rangle = \sum S_\mu^i(z) g^{\mu\nu} \oint \langle\phi_\nu, \frac{\phi_\alpha}{w-c}\rangle\ \frac{(\mathbf{t}^\alpha(w) - \delta^\alpha w)}{2\pi i(z+w)}\ dw.$$

Computing the countur integral we arrive at the formula

$$\frac{e^{u^i/z}}{\sqrt{\mathbf{D}_i}}(z+\sum \mathbf{T}^i_k(-z)^k) = \sum_{\mu\nu} S^i_\mu(z)_{t=t(\mathbf{t})} g^{\mu\nu} \times \{\langle \phi_\nu, 1, \mathbf{t}(c)-c\rangle +$$
$$(-z)\langle \phi_\nu, 1, 1, \mathbf{t}(c)-c\rangle + (-z)^2\langle \phi_\nu, 1, 1, 1, \mathbf{t}(c)-c\rangle + ...\}_{t=t(\mathbf{t})}. \tag{24}$$

Here $\langle \phi_\nu, 1, ..., 1, f(c)\rangle(t)$ coincide with multiple t-derivatives of $\langle \phi_\nu, f(c)\rangle(t)$ in the direction of the vector 1. Here the notation $S^i_\mu(z) = e^{u^i/z}(\sum (R_k)^i_j z^k)\Psi^j_\mu$ in the context of localization formulas should eventually refer to ${}^{new}S$, the fundamental solution matrix modified by the Bernoulli constants.

We take (23) and (24) on the role of definitions for $\mathbf{V}^{ij}_{kl}, \mathbf{T}^i_k$ *and* $\mathbf{D}_i$ *in the formula (22).*

By definition $\mathbf{T}^i_1 = 0$ while $\mathbf{T}^i_0 = 0$ follows from the criticality condition in (19). It is straightforward to check that the definition reduces to (6,7) when $t_1 = t_2 = ... = 0$. With these definitions in force, and the compensating constants in $R(z)$ in place, we arrive at the following theorem.

Theorem 2. *The formula (22) for higher genus descendent potentials holds true in equivariant GW-theory of Hamiltonian tori actions with isolated fixed points.*

Example 7. We compute from (24) that

$$\mathbf{D}_i^{-1/2}(\mathbf{t}) = \Delta_i^{-1/2}[\sum \frac{\partial u^i}{\partial t^\mu} g^{\mu\nu}\langle \phi_\nu, 1, 1, c-\mathbf{t}(c)\rangle](t(\mathbf{t})).$$

Along the lines of Example 6 we get $d\mathcal{F}^1 =$

$$= \sum(\frac{\mathbf{V}^{ii}_{00}}{2}du_i + \frac{d\mathbf{D}_i}{48\mathbf{D}_i}) = d\{F^1(t(\mathbf{t})) - \frac{1}{24}\ln[\prod_i \frac{\partial u^i}{\partial t^\mu} g^{\mu\nu}\langle \phi_\nu, 1, 1, c-\mathbf{t}(c)\rangle]\}.$$

This answer actually coincides with the well-known result [5]

$$\mathcal{F}^1(\mathbf{t}) = F^1(t(\mathbf{t})) + \frac{1}{24}\ln\det[\frac{\partial t^\mu}{\partial t^\nu_0}].$$

Indeed, differentiating the criticality condition $\langle \phi_\delta, 1, c-\mathbf{t}(c)\rangle = 0$ in (19) we find that $g^{\alpha\varepsilon}\langle \phi_\varepsilon, \phi_\beta, 1, c-\mathbf{t}(c)\rangle(t(\mathbf{t}))$ form the matrix inverse to $[\partial t^\mu/\partial t^\nu_0]$. On the other hand, the genus 0 topological recursion relation (or WDVV-equation) implies

$$g^{\alpha\varepsilon}\langle \phi_\varepsilon, \phi_\beta, 1, c-\mathbf{t}(c)\rangle = g^{\alpha\varepsilon}\langle \phi_\varepsilon, \phi_\beta, \phi_\mu\rangle g^{\mu\nu}\langle \phi_\nu, 1, 1, c-\mathbf{t}(c)\rangle.$$

In other words, $[\partial t/\partial t_0]^{-1}$ coincides with the linear combination with coefficients $g^{\mu\nu}\langle \phi_\nu, 1, 1, c-\mathbf{t}(c)\rangle$ of the commuting matrices $[g^{\alpha\varepsilon}F^0_{\varepsilon\mu\beta}]$ of quantum multiplication operators $\phi_\mu\bullet$. Thus the eigenvalues of the matrix are the linear combinations $(\partial u^i/\partial t^\mu)g^{\mu\nu}\langle \phi_\nu, 1, 1, c-\mathbf{t}(c)\rangle$, and the determinant is their product.

Example 8. Consider GW-theory with the target space $X = pt$. Then $u = t$, $\langle 1, 1/(z-c)\rangle = \exp(t/z)$ and respectively $\Delta = 1$ and $V_{kl} = 0$. We find the RHS in (22) equal to $\tau(\hbar\mathbf{D}; \mathbf{T})$ with $\mathbf{D}$ and $\mathbf{T}_k$ computed as follows. We have $f(t; \mathbf{t}) := \langle 1, 1, \mathbf{t}(c)-c\rangle(t) = \sum t_k t^k/k! - t$. The relation (19) turns into $f(t(\mathbf{t}); \mathbf{t}) = 0$, while $\mathbf{D}^{-1/2} = -f'(t(\mathbf{t}); \mathbf{t})$, and

$$\mathbf{T}_1 - 1 = f'(t(\mathbf{t}); \mathbf{t})\sqrt{\mathbf{D}},\ \mathbf{T}_2 = f''(t(\mathbf{t}), \mathbf{t})\sqrt{\mathbf{D}},\ \mathbf{T}_3 = f'''(t(\mathbf{t}); \mathbf{t})\sqrt{\mathbf{D}}, ...$$

Note that $(t_k - \delta_{k,1}) \mapsto \partial^k f(t; \mathbf{t})/\partial t^k$ is the string flow on the curve space so that $\mathbf{T}$ is obtained from $\mathbf{t}$ by applying the string flow until $t_0 = 0$ and then applying the dilaton flow until $t_1 = 0$. The potentials $\mathcal{F}^g_{pt}(\mathbf{t})$ with $g = 0, 1$ vanish when $t_0 = t_1 = 0$, and for $g \geq 2$ are preserved by the string flow and are homogeneous of

degree $2-2g$ with respect to the dilaton flow. We conclude that indeed $\tau(\hbar\mathbf{D};\mathbf{T})$ coincides with $\exp\sum_{g\geq 2}\hbar^{g-1}\mathcal{F}^g_{pt}(\mathbf{t})$.

Finally, the formulas (22–24) agree with our Lemma about Hodge intersection numbers in the following sense: the Lemma follows formally from our claim that, in the current setting with descendents as well, the s-deformation (13) of (2) is compensated by the modification $\exp(u/z)\mapsto\exp(u/z+B_2s_1z/2!+B_4s_2z^3/4!+...)$ described in the part (c) of Proposition.

Conjecture 2. *With the notations (2,23,24) in force, the formula (22) represents higher genus gravitational descendents of compact symplectic manifolds with generically semisimple quantum cup-product.*

Remark. Generally speaking, the task of deriving Conjectures 1 and 2 from Theorems 1 and 2 by passing to the non-equivariant limit is open and non-trivial. We will show in [17] how to do this in the case of complex projective spaces and their products.

3.3. **Concluding remarks.** The proposal (1,22) should be exposed to further tests. We expect it to be consistent with any relations in cohomology of Deligne–Mumford spaces (see [12, 13] for some such ralations found in genus 0 and 2). In fact we hope that our Theorems 1 and 2 on equivariant GW-potentials impose some constraints on topology of Deligne–Mumford spaces so tight that the corresponding results in abstract semisimple GW-theory would follow. Respectively, it should be interesting to make such constraints as explicit as possible and to understand better the geometrical structure on Frobenius manifolds encrypted by (1) and (22). To this end, we should say that the formulas (1,22) can be rewritten differently. Using the Fourier transform they can be given a form of path integrals. Substituting matrix Airy integrals for the Kontsevich – Witten function (2) we can relate the formulas to multi-matrix models. Perhaps, the most useful is the representation-theoretic formulation [17], which automatically restores the (descendent) potentials of genus 0 and 1 and yields a formula for the complete tau-function $\exp[\sum_{g\geq 0}h^{g-1}\mathcal{F}^g]$. It enables us to show (see [17]) that the proposal agrees with the Virasoro constraints [8] and to derive the "Virasoro conjecture" for GW-invariants of complex projective spaces and their products. We also expect the formula for the tau-function to be helpful in the construction of the bihamiltonian structure of the KdV-like integrable hierarchy whose approximations are studied in [6, 7, 8, 9].

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